Task 3.2 Final Report

Formulation of numerical methods for fractal structures

Abstract:

This task is essentially aimed at analyzing critically the problems associated with the numerical simulations of fractal structures based on the results achieved in Tasks 2.1 and 2.2, and at developing efficient computational approaches for a reliable numerical simulation of high-order iteration prefractal wire antennas using the Electric Field Integral Equation (EFIE).

Keyword list: Numerical simulations, wire antennas, moment method, thin-wire approximation.
FORMULATION OF NUMERICAL METHODS FOR FRACTAL STRUCTURES

1. INTRODUCTION

This task is essentially aimed at analyzing critically the problems associated with the numerical simulations of fractal structures based on the results achieved in Tasks 2.1 and 2.2, and at developing efficient computational approaches for a reliable numerical simulation of high-order iteration prefractal wire antennas using the Electric Field Integral Equation (EFIE). It was agreed in the Kick-off meeting that the research would be concentrated in wire antennas, that can be either a cylindrical wire or a strip.

Three main items have been addressed:

2. VALIDITY OF THE THIN-WIRE APPROXIMATION

(Report - WP3 T3.2 UPC T0+12 Prefractal wire modeling)

The accuracy of the commonly used approximations for the analysis of thin-wire antennas when applied to fractal structures has been evaluated. It is concluded that:

- The thin-wire kernel of the Electric Field Integral Equation (EFIE) can be used in low-iteration pre-fractals without introducing significant errors. Examples of computer codes based on thin-wire approximation that have been used in this project are: DOTIG (time domain), NEC (frequency domain) and FIESTA (frequency domain).

- When the EFIE is discretized by Method of Moments in subdomain basis functions, the thin-wire kernel with or without equivalent radius produces important errors in the computation of highly iterated pre-fractal antenna parameters, since the pre-fractal curve segments become comparable to the wire diameter. Even the full kernel fails because the assumption of constant current along the wire circumference is no longer valid and the wire cylindrical segments overlap at corners. An extrusion strip can be used in this case in order to obtain an accurate subdomain discretization of the EFIE.

- The results of extrusion strip models computed with FIESTA code have been compared with the simpler thin-wire models. The comparison results fully support the conclusions above.
3. IMPLEMENTATION OF WIRE ANTENNA ANALYSIS IN FIESTA COMPUTER CODE

(Report - WP3 T3.2 UPC T0+12 Wire modelling in FIESTA)

The computer codes for the analysis of wire antennas available at the start of FractalComs project, namely DOTIG and NEC, do not use advanced techniques for the solution of very large systems of equations. On the other hand, FIESTA code is able to solve the EFIE with hundreds of thousands of unknowns, but at the project start date only the analysis of perfectly conducting surfaces was implemented.

In order to solve very large wire antenna problems, the Electric Field Integral Equation for wires has been implemented in FIESTA. Three different formulations of the kernel have been programmed: thin-wire, thin-wire with equivalent radius and full kernel.

4. NEW APPROACHES FOR SOLVING THE EFIE OF WIRE ANTENNA ANALYSIS

(Report - WP3 T3.2 ROME T0+12)

Some numerical approaches based on the Galërkin-Petrov expansion for solving the EFIE in pre-fractal structures have been studied. The approach proposed proves to be very versatile and computationally efficient. Particular attention is oriented towards the analysis of the thin-wire approximation from the numerical point of view, by discussing critically its validity and pitfalls, in the light of the compactness property of the associated linear operator. Numerical simulations have been performed on different pre-fractal structures, highlighting some interesting properties of the current profiles.

It is clear from the analysis developed in WP2 that any numerical simulation of the EFIE equations on fractal structures cannot be conveniently grounded on point-matching approaches, but should be framed within a weak-formulation of the EFIE resulting from a Galërkin-Petrov expansion in a basis of entire-domain functions. The proposed technique is therefore based in the Method of Moments Galërkin discretization of the Pocklington thin-wire kernel in a set of complete-domain sinusoidal basis functions.

Although the Pocklington equation for dipolar antennas has been known and widely used for a long time, the computational issues associated with the analysis of thin wires are still an open problem. Recent articles conclude that the thin-wire approximation gives rise to an “ill-posed problem”, and that all the simulations based on this approach are scientifically unreliable.

Since the thin-wire approximation is one of most useful simplifying assumptions in applied electromagnetism, we have analyzed very carefully this issue, and we have shown that, albeit the criticism of some authors is motivated by a correct mathematical observation, in practical problems involving thin-wire antennas the entire-domain Galërkin approach furnishes reliable and convergent approximations for the computation of the current and the antenna parameters.
Of course, the numerical problems are much more delicate in dealing with fractal wire antennas, since the geometric complexity of the structures is superimposed to the intrinsic difficulty of handling a singular integral equation.
On the modelling of pre-fractal wire antennas

Abstract:
This report studies the accuracy of the commonly used approximations for the analysis of thin-wire antennas when applied to fractal structures. It is concluded that the thin-wire kernel of the Electric Field Integral Equation can be used only in low-iteration pre-fractals, while highly iterated ones must be modelled as an extrusion strip.

Keyword list: Numerical simulation, wire models, thin-wire approximation, pre-fractal antennas.
RELATED WP AND TASKS (FROM THE PROJECT DESCRIPTION)

WP3: Software simulation tool

Task 3.2: Formulation of numerical methods for fractal structures

b) Develop new approximations for thin-wire antennas along fractal curves

c) Compare results in b) with classical numerical schemes pushed, if possible, to the limit of complexity afforded by the technology.

1 INTRODUCTION

Thin wire based models have a long history in numerical electromagnetics, for two reasons: Firstly, many electromagnetical devices have a wire-like geometry. In the most general sense, this means that they are built up of components of conducting material that are electrically large along a given curve, and very thin (\(<<\lambda\)) in cross section. Examples are dipole and monopole antennas, or arrays of dipoles, Yagi-Uda antennas, helices, etc. Secondly, under certain conditions, to be addressed in the following section, the wire model allows for approximations that greatly facilitate the numerical analysis.

Fractal antennas do, at first sight, comply with the definition of ‘wire-like’ as formulated above, because they are built up of long and thin elements of conducting material. However, typical of a fractal is the extreme curvedness of the elements, at all scales of magnification (see Fig.1.1). This may undermine the applicability of the approximations that make the strength of the thin wire model. In this report, the applicability of the wire model to fractal antennas is evaluated.

Fig. 1.1. Pre-fractal Koch curve.

2 THIN WIRE APPROXIMATION

The thin wire approximation assumes that the conducting element can be globally characterised by a (set of) defining curve(s) and locally by a cross-section that is

1. circular
2. has a radius \(a<<\lambda\).

Such that, defining a local \(z\)-axis tangent to the defining curve:
1. The surface current density circumferential component \( J_\phi \) can be neglected.

2. The surface current density axial component \( J_z \) is constant along the circumference.

3. The same points 1 and 2 apply to the excitation vector field.

As a consequence, the surface current density reduces to a total current \( I_z \) through the wire cross-section and the excitation vector field reduces to a potential along the defining curve.

These approximations, applied to the case of a straight thin wire, lead to the well known Pocklington equation [Balanis]:

\[
-j\omega E_z = \frac{1}{2\pi} \left( k^2 + \frac{d^2}{dz^2} \right) \int I_z(z')G(R)dz'
\]

(1)

where

\[ \text{(2)} \]

is the thin wire Green's function, with

\[
R = \sqrt{(z-z')^2 + (\rho - \rho' - a\cos\phi)^2 + (a\sin\phi)^2}
\]

(3)

The extension to curved wires basically consists in sub-dividing the defining curve into segments, short enough to be approximated by straight wires, and then to apply (1) with the provision that, for non-parallel wires, \( \frac{d^2}{dz^2} \rightarrow \frac{d^2}{dzdz'} \).

On every straight wire segment, the current is represented by a coefficient times a basis function. Some common basis functions are shown in Fig. 2.1.

Fig. 2.1. Examples of thin wire basis functions on a set of straight wire segments.
3 LIMITATIONS

3.1 Segment length versus wire radius

Equation (2), the Green's function for a circumferentially constant current density, is also referred to as the full wire kernel. It has to be integrated numerically for all the mutual impedances between all wire segments in a given model (only for the auto-impedance it has an analytical solution), and this can be exceedingly time consuming. Therefore, usually, further approximations are invoked (see Report - WP3 T3.2 UPC T0+12 Wire modelling in FIESTA). However, all these further approximations have in common, that the wire radius $a$ is not only subject to $a \ll \lambda$, but also to $a \ll \Delta$, where $\Delta$ is the segment length (see Fig. 3.1).

![Segment length versus wire radius](image)

This obviously limits their applicability to fractal structures.

Table 3.1 shows the restrictions on the different methods addressed in [Report - WP3 T3.2 UPC T0+12 Wire modelling in FIESTA].

<table>
<thead>
<tr>
<th>Method</th>
<th>Minimum segment length $\Delta$ $(a = \text{segment radius})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>thin wire kernel</td>
<td>$8a$ (1% error in current, NEC spec.)</td>
</tr>
<tr>
<td>NEC extended kernel</td>
<td>$2a$ (1% error in current, NEC spec.)</td>
</tr>
<tr>
<td>Imbriale equivalent radius</td>
<td>$a$ (numerical evidence [ref], see also Fig. 3.2)</td>
</tr>
<tr>
<td>Full wire kernel</td>
<td>no restriction</td>
</tr>
</tbody>
</table>

Fig. 3.2 illustrates the difference between the equivalent radius method and the full wire kernel for $\Delta = 2a$.

![Input impedance](image)
Table 3.1 indicates the limit, using the various thin wire methods, to the detail of the model structure, in other words the order (number of fractal iterations) of the pre-fractal model to be analysed. Using the full wire kernel, there is no limit with regard to the $\Delta/a$ ratio.

### 3.2 Mutually close wire segments

Another problem with highly iterated fractal structure is the occurrence of mutually very close wire segments that do not lay on the same straight line. If the mutual distance between the wire axes is of the same order as the wire radius, then the assumption that the current is constant along the circumference does not hold anymore, because of the interaction between the two segments. This is illustrated for the case of two parallel wires with a simple experiment shown in Fig. 3.3.

Of course this is true for any wire structure with bends or corners, but if the segment $\Delta/a$ is sufficiently large, and the wires do not run parallel in close proximity as they do in Fig. 3.4, the error only occurs near the end-points of the segments, and will be small compared to the total interaction term. However, in highly iterated fractal structures, *every* point on the wire is typically in close proximity to other non-collinear wires.

This leads to the conclusion that any thin wire model will fail for highly iterated fractals, and that the limit is not determined by $\Delta/a$ but rather by the ratio between the wire thickness and the length of the straight wire elements at the basis of the pre-fractal structure, $L >> a$, as illustrated in Fig. 3.3.

![Fig. 3.3: Four-iteration Koch curve, with elementary wires of length L](image-url)
3.3 Corner modeling

Another issue of importance in the application of a thin wire model to a fractal structure is the treatment of corners or wire junctions. In all the methods summed up in table 2.1 and explained in [Report - WP3 T3.2 UPC T0+12 Wire modelling in FIESTA], the segments are considered to be cylindrical tubes with flat end-faces. This means that a connection between two non-parallel segments gives rise to a situation as depicted in Fig. 3.5
Consequently, the surface current, which is defined on the whole cylindrical surface (excluding the endfaces, since only the axial component is considered), will, in situations like the one in Fig. 3.5, inevitably fall inside the adjacent segment.

Numerically, this effect is commonly ignored (NEC, FIESTA), which is unlikely to lead to numerical problems, since in general, only a few abscissa are needed on the wire surfaces and it is easy to prevent them from coinciding. Again, like with the problem addressed in the previous section, in the case of long thin segments \((a \ll \Delta)\), the inaccuracy caused by ignoring this effect is small. However, when \(a \rightarrow \Delta\), one only needs to imagine the situation of Fig. 3.5 in the pre-fractal wire of Fig. 3.3 to see that treating the wire segments as if unaffected by adjacent segments is unrealistic.

4 NUMERICAL VALIDATION COMPARED WITH STRIPS

The surface formulation of the Electric Field Integral Equation is free from the inaccuracies that arise from the thin-wire model. Therefore, if the wire is modeled as a strip, that can be easily meshed in triangle, rooftop or quadrangular basis functions, the result is a accurate MoM discretization if enough integration points are used on each triangle [D6 Final report Task 3.1].

The strip configuration that more can model accurately a wire is a extrusion strip, obtained by extrusion of the pre-fractal curve in the direction perpendicular to the plane containing the curve (Fig. 4.1).

![Fig. 4.1. One iteration extrusion-strip Koch antenna, discretized in triangular patches.](image)

The conventional planar strip configuration has been discarded for comparison with the wire model, because the width of the strip interferes with the strip angles, while in extrusion-strips it does not. As a consequence, the geometry of a highly iterated planar strip loses the auto scaling property of pre-fractals, while extrusion-strip preserve it.

Table 4.1 shows the resonant frequency (GHz) of wire, planar-strip and extrusion-strip Koch monopoles. Height is 6cm and the strip width 1mm or 0.5mm. The wire models have a radius equal to the strip width over \(\pi\).

Table 4.1

<table>
<thead>
<tr>
<th></th>
<th>Wire 1mm</th>
<th>Wire 0.5mm</th>
<th>Planar 1mm</th>
<th>Planar 0.5mm</th>
<th>Extrusion 1mm</th>
<th>Extrusion 0.5mm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency (GHz)</td>
<td>2.4</td>
<td>2.6</td>
<td>2.8</td>
<td>3.0</td>
<td>3.2</td>
<td>3.4</td>
</tr>
</tbody>
</table>

Two computer software codes have been used to analyze the wire model: NEC-2 [Burke] and FIESTA; with three different integral equation kernel formulations: thin-
wire approximation, extended kernel and thin-wire with equivalent wire radius [Imbrialle].

**RESONANT FREQUENCIES (GHz)**

<table>
<thead>
<tr>
<th>Software</th>
<th>NEC wire</th>
<th>NEC wire</th>
<th>FIESTA wire</th>
<th>FIESTA wire</th>
<th>FIESTA strip</th>
<th>FIESTA strip</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model</td>
<td>thin-wire kernel</td>
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<td>thin-wire kernel</td>
<td>eq. radius kernel</td>
<td>planar strip</td>
<td>extrusion strip</td>
</tr>
<tr>
<td>Radius or width</td>
<td>0.5/π (mm)</td>
<td>0.5/π (mm)</td>
<td>0.5/π (mm)</td>
<td>0.5/π (mm)</td>
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<td>0.5 mm</td>
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<tr>
<td>K-0</td>
<td>1.202</td>
<td>1.202</td>
<td>1.238</td>
<td>1.238</td>
<td>1.195</td>
<td>1.196</td>
</tr>
<tr>
<td>K-1</td>
<td>0.987</td>
<td>0.987</td>
<td>0.993</td>
<td>0.993</td>
<td>0.988</td>
<td>0.990</td>
</tr>
<tr>
<td>K-2</td>
<td>0.866</td>
<td>0.866</td>
<td>0.864</td>
<td>0.863</td>
<td>0.858</td>
<td>0.859</td>
</tr>
<tr>
<td>K-3</td>
<td>0.799</td>
<td>0.799</td>
<td>0.785</td>
<td>0.783</td>
<td>0.787</td>
<td>0.783</td>
</tr>
<tr>
<td>K-4</td>
<td>0.773</td>
<td>0.773</td>
<td>0.772</td>
<td>0.763</td>
<td>0.762</td>
<td>0.750</td>
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<tr>
<td>K-5</td>
<td>0.782</td>
<td>0.758</td>
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<td>1/π (mm)</td>
<td>1/π (mm)</td>
<td>1 mm</td>
<td>1 mm</td>
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<tr>
<td>K-0</td>
<td>1.194</td>
<td>1.194</td>
<td>1.232</td>
<td>1.232</td>
<td>1.185</td>
<td>1.188</td>
</tr>
<tr>
<td>K-1</td>
<td>0.993</td>
<td>0.993</td>
<td>1.001</td>
<td>1.001</td>
<td>0.994</td>
<td>0.997</td>
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<tr>
<td>K-2</td>
<td>0.890</td>
<td>0.890</td>
<td>0.875</td>
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<td>0.880</td>
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<tr>
<td>K-3</td>
<td>0.842</td>
<td>0.842</td>
<td>0.833</td>
<td>0.828</td>
<td>0.828</td>
<td>0.821</td>
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<tr>
<td>K-4</td>
<td>0.831</td>
<td>0.832</td>
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<td>0.799</td>
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<tr>
<td>K-5</td>
<td>0.877</td>
<td>0.774</td>
<td></td>
<td></td>
<td></td>
<td>0.788</td>
</tr>
</tbody>
</table>

Table 4.1: Resonant frequency of wire and strip models of the Koch antenna.

Table 4.2 shows the relative errors in the resonant frequency computation taking as a reference the results of the extrusion-strip model.

The results in tables 4.1 and 4.2 lead to the following conclusions, that corroborate the theory of the preceding sections:

- The wire models error grows rapidly after iteration K3. For three and four iterations, the pre-fractal curve segment length is respectively 2.2mm or 0.75mm, comparable to the wire diameter, and therefore the thin-wire approximation fails.
- When the thin-wire approximation fails, the error is smaller for the thinner wire. The error in the K4 1mm-strip is 3.1% and in the 0.5mm-strip 1.8%.
- The error with the equivalent radius model of Imbrialle is smaller than with the plain thin-wire approximation when the segment length becomes comparable to the wire diameter.
- For low-iteration pre-fractals, all models (wire, planar strip and extrusion strip) give the same results.
ERROR WITH RESPECT TO EXTRUSION STRIP

<table>
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<tr>
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<td>0.5/p (mm)</td>
<td>0.5 mm</td>
<td>0.5 mm</td>
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<tr>
<td>K-0</td>
<td>0.53%</td>
<td>0.53%</td>
<td>3.54%</td>
<td>3.54%</td>
<td>-0.06%</td>
<td>0.00%</td>
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<tr>
<td>K-1</td>
<td>-0.27%</td>
<td>-0.27%</td>
<td>0.31%</td>
<td>0.30%</td>
<td>-0.19%</td>
<td>0.00%</td>
</tr>
<tr>
<td>K-2</td>
<td>0.78%</td>
<td>0.79%</td>
<td>0.61%</td>
<td>0.49%</td>
<td>-0.12%</td>
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<td>K-3</td>
<td>2.02%</td>
<td>2.03%</td>
<td>0.18%</td>
<td>-0.04%</td>
<td>0.46%</td>
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<tr>
<td>K-4</td>
<td>3.04%</td>
<td>3.06%</td>
<td>2.92%</td>
<td>1.76%</td>
<td>1.59%</td>
<td>0.00%</td>
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<tr>
<td>K-5</td>
<td></td>
<td></td>
<td>6.28%</td>
<td>2.93%</td>
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<td>K-0</td>
<td>0.47%</td>
<td>0.47%</td>
<td>3.69%</td>
<td>3.70%</td>
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<td>-0.37%</td>
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<td>0.42%</td>
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<td>K-2</td>
<td>1.05%</td>
<td>1.08%</td>
<td>-0.60%</td>
<td>-0.69%</td>
<td>-0.07%</td>
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<td>K-3</td>
<td>2.55%</td>
<td>2.49%</td>
<td>1.40%</td>
<td>0.80%</td>
<td>0.82%</td>
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<tr>
<td>K-4</td>
<td>4.00%</td>
<td>4.11%</td>
<td>8.48%</td>
<td>3.10%</td>
<td>2.70%</td>
<td>0.00%</td>
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<tr>
<td>K-5</td>
<td></td>
<td></td>
<td>11.35%</td>
<td>-1.74%</td>
<td></td>
<td>0.00%</td>
</tr>
</tbody>
</table>

Table 4.2: Error in the resonant frequency computation with respect to the extrusion-strip model.
5 CONCLUSIONS

Based on the findings in section 3 and 4, the following conclusions can be drawn regarding the use of the thin wire model for the analysis of pre-fractal antennas:

- An important parameter in thin straight wire models is the ratio of segment length and segment radius (∆/a). For ∆/a>1, fast approximative methods can be used, for ∆/a<1, one needs to revert to the more computationally demanding full wire kernel.

- However, in pre-fractal structures the occurrence of many sharp bends (corners) limits this ratio to ∆/a>>1; and the error introduced by mutually close non-collinear segments restricts the minimum length L of the pre-fractal model wire segments to L>>a. In both cases the assumption of constant current along the wire circumference fails.

- Summarising the above: The thin wire model can only be used with confidence for modelling pre-fractal structures of a sufficiently low order (iteration) that L>>a. Then, the first criterion, ∆/a>>1, can easily be fulfilled by using one segment per fractal building block.

- The best way to accurately model highly iterated pre-fractal curves is a extrusion-strip model rather than a planar strip or a thin wire. For that reason, most of the simulations in WP3 aimed at drawing conclusions for WP1 will be made with extrusion-strip models.

- However, wire models are still useful for analysing low-order pre-fractals, and therefore NEC, FIESTA and the time-domain DOTIG code will be extensively used in this project.
BIBLIOGRAPHY


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Abstract:

This report describes the extension to thin wire models of the UPC in-house Method of Moments computer program FIESTA (Fast Integral Equation Solver for scatterers and Antennas in 3-D). The reasons for developing a new thin wire code instead of using a common, well established existing code such as NEC (Numerical Electromagnetics Code) are addressed. On the one hand, the FIESTA implementation allows for greater flexibility in the geometrical model definition, on the other hand FIESTA incorporates some advanced solution algorithms that enable the solution of very large problems in terms of the number of unknowns. Both reasons are of particular interest when it comes to accurate modeling of fractal antennas. The theory behind the FIESTA thin wire code is presented and compared with several common approaches in the literature. A number of numerical experiments are presented that validate the FIESTA code by comparison with results from NEC and theoretical predictions.
# Wire Modeling in FIESTA

**Theory & Validation**

January 23, 2003

## Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>2.1</td>
<td>3</td>
</tr>
<tr>
<td>2.2</td>
<td>3</td>
</tr>
<tr>
<td>2.3</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>3.1</td>
<td>8</td>
</tr>
<tr>
<td>3.2</td>
<td>9</td>
</tr>
<tr>
<td>3.3</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
</tr>
<tr>
<td>5</td>
<td>12</td>
</tr>
</tbody>
</table>
1 Introduction

A key factor of a successful Method of Moments computer code is the generality and versatility of the geometrical model on which the code is based. Surface patch models, like the one introduced by Rao, Wilton and Glisson in 1982 [5], have replaced the once common wire grid models, when it comes to modeling perfectly conducting surfaces. However, the capacity to model thin wire structures remains important in numerical electromagnetics (dipoles, loops, wire antennas) and although one can imagine modeling a wire as a thin strip of surface patches, this is neither very efficient nor very accurate. It is therefore desirable for a MoM computer code to be able to deal with both models, preferably combined. An example of such a code is the widely used computer program for electromagnetic simulations NEC (Numerical Electromagnetics Code [7]).

This document describes the wire modeling function that has been added to the surface patch based program FIESTA, developed at EEF/UPC. The reasons for adding this function, instead of relying on NEC for problems involving wires are:

1. FIESTA includes two fast MoM solution methods, the MLFMA [3] and the MLMDA [4], which allow for the solution of problems with a number of unknowns that is several orders of magnitude higher than NEC, which only incorporates solution by direct impedance matrix inversion.

2. FIESTA gives the choice to solve a patch model using the EFIE, which allows for open structures to be modeled with patches. NEC on the other hand uses the MFIE on the patched surfaces and the EFIE on the wire structures. The MFIE cannot be applied to open surfaces. An example of an open surface in combination with a wire structure is a parabolic reflector fed by a small wire antenna.

3. The thin wire model involves some approximations (explained in detail in chapter. 2) that are valid only if the radius of the wires is much smaller than the wavelength. In addition, the wire model implementation in NEC requires the radius of the wires to be small compared to the length of the straight wire segments that represent the geometry. In FIESTA a different algorithm is used that is valid for arbitrarily short segments.

Points number one and three are of particular interest in the light of an eventual application to the analysis of fractal wire antennas: The MLFMA/MLMDA are particularly suitable to problems with a two dimensional geometry (surfaces). Then, their computational complexity is $N \log N$ as opposed to $N^3$ for the direct solution (where $N$ is the number of unknowns involved). In general, this is not the case for wires, which have a geometrical dimension of one. But fractals have a geometrical dimension greater than one (they 'fill up' the space), which is favorable for the fast methods. Point three is important because fractals are never smooth. Consequently, modeling them accurately may require many short segments.
The theory behind the wire functionality in FIESTA is explained in chapter 2. Chapter 4 presents several numerical experiments validating the FIESTA code against results obtained by NEC and against theoretical results.

2 Theory

2.1 EFIE

The method of Moments solution of radiation and scattering problems for Perfectly Conducting (PEC) objects, is based on a formulation of the problem in terms of an Integral Equation (IE). The IE is derived by enforcing a boundary condition on the surface of the PEC objects, either in terms of the electric field, leading to the Electric Field Integral Equation (EFIE) or in terms of the magnetic field, leading to the Magnetic Field Integral Equation (MFIE). The MFIE numerically fails for electrically thin geometries, so the thin wire formulation is necessarily based upon the EFIE. The EFIE is given by

\[ \hat{n} \times \vec{E}^i(\vec{r}) = \hat{n} \times \frac{j\eta}{4\pi k} (\vec{A}(\vec{r}) + \nabla \Phi(\vec{r})) \],

in which \( \vec{r} \) is any point on the object surface, \( \hat{n} \) is the outward surface normal at that point, \( \vec{E}^i \) is the imposed incident electric field,

\[ \vec{A}(\vec{r}) = k^2 \int_S \frac{e^{-jkR}}{R} \sigma \cdot ds, \quad R = |\vec{r} - \vec{r}'|, \]

the vector potential associated with the surface current distribution \( \vec{J} \) over the entire problem surface \( S \), and

\[ \Phi(\vec{r}) = \int_S \frac{e^{-jkR}}{R} \sigma \cdot ds, \]

the scalar potential, associated with the surface charge density \( \sigma \).

2.2 Thin wire approximation

In the case were the surface \( S \) consists of electrically thin wires, some approximations can be made in \( (1) \):

- The current density on the wire surface has no circumferential component and is constant over the wire surface along \( \phi \). Consequently we can express it as

\[ \vec{J}(z, \phi) = \frac{1}{2\pi a} I_z(z) \hat{u}_z \]

where \( a \) is the wire radius and \( I_z(z) \) is the current per unit-length on the wire. The \( z \)-coordinate can be interpreted as a parameter along the (not necessarily straight) wire axis. The unit vector \( \hat{u}_z \) is the local tangent
to this axis. In a numerical model, the geometry will generally be repre-
represented by piece-wise straight segments. Furthermore, the surface charge
distribution on the wire can be expressed as
\[
\sigma = \frac{j}{\omega} \nabla \cdot \mathbf{J} = \frac{j}{2\pi \omega a} \frac{dI_z(z)}{dz} = \frac{1}{2\pi a} q(z)
\]  
(5)
where \(q(z)\) is the charge per unit-length on the wire.

- The incident electric field, \(\mathbf{E}^i\) is constant along the wire circumference. Furthermore, the angular component of \(\mathbf{E}^i\) is negligible. The incident field on the wire surface is therefore expressed as \(E^\text{inc}_z(z)\).

With these assumptions, (2) reduces to
\[
\mathbf{A} (\mathbf{r}) = k^2 \int_S I_z(z') G(R) \hat{u}_z' d\mathbf{z}',
\]  
(6)
and (3) reduces to
\[
\Phi (\mathbf{r}) = \int_S q(z') G(R) dz'
\]  
(7)
where \(G(R)\) is a Greens function incorporating the circumferential integration, given by
\[
G(R) = \int_0^{2\pi} \frac{1}{2\pi} \frac{e^{-jkR}}{R} d\phi', \quad R = |\mathbf{r} - \mathbf{r}'|.
\]  
(8)

Pocklington’s well known integro-differential equation \([6]\) follows directly from the above formulation, when applied to the problem of the self impedance of one straight wire segment \(w\) aligned with the global \(z\)-axis, extending between \(z = z_1\) and \(z = z_2\), leading to
\[
E^\text{inc}_z(z) = \frac{j\eta}{4\pi k} \int_{z_1}^{z_2} I_z(z') \left\{ \frac{\partial^2}{\partial z^2} + k^2 \right\} G(R) dz'.
\]  
(9)
The above approximations are valid for \(a << \lambda\). For even smaller radiuses, a further approximation can be made, which is to replace the current by an infinitely thin filament of current on the wire axis. This reduces the Green’s function to the so called thin-wire kernel. In cylindrical coordinates, for wire segment \(w\), it is given by
\[
G_{\text{thin}}(z) = \frac{e^{-jk\sqrt{\rho^2 + (z-z')^2}}}{\sqrt{\rho^2 + (z-z')^2}}
\]  
(10)
The singularity that occurs in the self impedance evaluation becomes uninte-
grable with this approximation. A common approach to circumvent this problem, that yields good results for \(a\) sufficiently small \([11]\), is to evaluate the field at the wire surface \(\rho = a\), which keeps the kernel finite. Using the thin wire
kernel for mutual impedances of wire segments at a mutual (minimum) distance \(d >> a\), is always permitted. Only for the self- and very nearby mutual impedances the approximation fails for too large radii.

Tijhuis and Rubio [9] have shown that assuming the current to be concentrated on the wire axis (or, which leads to equivalent results, by assuming the current to be located on the wire surface, and evaluating the field on the axis), any circumferential variation in the incident field (surface current), as well as any azimuthal field (current) components, are automatically eliminated from the integral equation.

Imbriale [8] presents an extensive numerical investigation of the validity of the thin wire kernel as opposed to circumferential integration. His experiments show that the thin wire kernel validity not only depends on \(a/\lambda\), but also on \(a/\Delta\), where \(\Delta\) is the wire segment length. When \(\Delta\) and \(a\) are of comparable size, the thin wire kernel approach as presented above, with a field evaluation at \(\rho = a\) fails. Imbriale derives a \(\Delta\)-dependent equivalent radius, \(a_e\) to be used in the thin wire kernel instead of \(a\) (which can be interpreted as a ‘mean value’ of \(R\) along the circumference), and shows it to be superior for \(a/\Delta\) approaching unity. The equivalent radius is given by

\[
a_e = a(1 - 0.40976 \frac{a}{\Delta}). \tag{11}
\]

Burke and Poggio [7], the principal authors of the NEC computer code, adopt a different approach. It incorporates the thin wire kernel for small radii, citing numerical experiments that indicate the error in the surface current does not exceed 1\% if for all wire segments in the model \(\Delta > 8a\) (and \(\Delta \lesssim 0.1\lambda\)). For thicker wires, an extended thin wire kernel is incorporated. In this model, the current is assumed to be uniformly distributed around the circumference of the wire, and a series expansion around the wire axis is derived for the field, in terms of \((ka)^2\). The zero order term yields the thin wire kernel, the first order term yields the extended thin wire kernel. This kernel, according to cited numerical experiments, yields an error in the current smaller than 1\% for \(\Delta > 2a\).

### 2.3 Thin wires in FIESTA

For the implementation in FIESTA, a different approach was adopted. The problem geometry is discretised into electrically small straight wire segments. The surface current is represented by rooftop basis functions defined on every pair of two adjacent segments, as described in Sec.3.1, such that at a node where \(M\) segments meet, \(1 + 2 + \cdots (M - 1)\) basis functions are defined. The current on every segment is linear in \(z\), where \(z\) is a local coordinate defined along the segment, and, according to (5), the line-charge is constant on a segment. As in NEC, there are two options for the impedance calculations, a thin wire kernel and an extended wire kernel. In both cases, the factor \(e^{-jkR}/R\) in the kernel is rewritten as

\[
\frac{e^{-jkR}}{R} = 1 - \frac{e^{-jkR}}{R}. \tag{12}
\]
The second term in (12) is a smooth function of $R$ everywhere, which can be integrated numerically with very few integration points, independent of $a/\Delta$, as long as $a << \lambda$ and $\Delta << \lambda$. The $1/R$ term, which is strongly dominant for the self and near mutual impedances is dealt with differently for the thin wire kernel and the extended kernel.

- **Thin wire kernel:**
  For the case of the thin wire kernel, the expression for a constant integrand (the scalar potential in FIESTA) takes the general form
  \[ I_0 = \int_{z_1}^{z_2} \frac{1}{\sqrt{\rho^2 + z^2}} \, dz \]
  which has the closed form solution
  \[ I_0 = \ln \left( \frac{z_2 + \sqrt{z_2^2 + \rho^2}}{z_1 + \sqrt{z_1^2 + \rho^2}} \right). \]
  \[ (14) \]
  The above formulation was proposed by Gimmersky et al in [11] and applied successfully to the Method of Moment analysis of a center fed straight wire antenna. The expression for a linear integrand, also needed in FIESTA (for the vector potential)
  \[ I_1 = \int_{z_1}^{z_2} \frac{z}{\sqrt{\rho^2 + z^2}} \, dz \]
  has the closed form solution
  \[ I_1 = \left[ \sqrt{\rho^2 + z^2} \right]_{z_1}^{z_2} \]
  \[ (16) \]
  as readily verified by differentiating (16). Expressions (14) and (16) are also applicable to the self impedance, setting $\rho = a$.

- **Extended kernel**
  In the extended kernel model, the current and the incident field are assumed to be constant along the wire circumference. The second term on the LHS of (12) is again computed with a low order numerical integration rule in all cases (self and mutual impedance). The $1/R$ term is treated separately for three distinct cases:

  1. *‘Far’ segments (minimum mutual distance $d >> \Delta, a$).*
     These are treated the same as the thin wire kernel.
  2. *Singular kernel (self impedance, touching segments)*
     For the singular extended kernel, that is, current and charge are constant along the circumference and the field- or observation point $\vec{r}$ is located in the domain of integration, FIESTA uses closed series
solutions given by Butler in [10]. For the charge (constant along the axis), the integral over one segment of length $\Delta$, with the observation point at one end of the segment (at $z = 0$), has the form

$$\Im_0 = \frac{1}{\pi} \int_0^{\Delta} dz \int_0^\pi \frac{1}{R} d\phi,$$

which has the series solution

$$\Im_0 = \ln \frac{2\Delta}{a} + \sum_{p=1}^{\infty} \left( \frac{1/2}{p} \right) (2p - 1) \frac{(2p - 1)!}{(p!)^2} \left( \frac{a}{\Delta} \right)^{2p}, \quad 2a < \Delta,$$

which only depends on the quotient $a/\Delta$, and converges very rapidly. For $a < \Delta$, the logarithmic term converges to the same value as (14). For $a \to \Delta/2$ the series converges to machine precision after five terms. For the current (linear along the axis), the integral has the form

$$\Im_1 = \frac{1}{\pi} \int_0^{\Delta} dz \int_0^\pi \frac{1}{R} d\phi,$$

which has the series solution

$$\Im_1 = -\frac{4}{\pi} a \frac{a}{\Delta} + \sum_{p=0}^{\infty} \left( \frac{1/2}{p} \right)^2 (-1)^{p+1} (2p - 1) \left( \frac{2a}{\Delta} \right)^{2p}, \quad 2a < \Delta,$$

which again only depends on the quotient $a/\Delta$. For $a <\Delta$, the infinite sum is negligible and the logarithmic term converges to the same value as (16) (apart from a normalisation factor $1/\Delta$ in (19)). For $a \to \Delta/2$ the series converges to machine precision after five terms. Both (18) and (20) can not be used for $a = \Delta/2$. However, FIESTA does allow for wires with $a \geq \Delta/2$. To find the correct values for the integrals for a wire with $\Delta = \Delta_s < 2a$, first $\Im_0(2a)$ is computed, as

$$\Im_0(2a) = \lim_{\varepsilon \to 0} \Im_0 [2a - \varepsilon].$$

Then,

$$\Im_0^{rcs} = \frac{1}{\pi} \int_{\Delta_s}^{2a} dz \int_0^\pi \frac{1}{R} d\phi$$

is computed numerically, and finally

$$\Im_0(\Delta_s) = \Im_0(2a) - \Im_0^{rcs}.$$
into a one dimensional integral. Interchanging the order of integration and expanding $R$

$$
\Im^\text{res}_0 = \frac{1}{\pi} \int_0^\pi \left\{ \int_{\Delta_s} \frac{1}{\sqrt{2^2 + 4a^2 \sin^2 \phi}} \, dz \right\} \, d\phi,
$$

(23)

the inner integral is recognised as being equal to (13). For $\Im^\text{res}_1$, (15) is used.

3. ‘Near’ segments (non-overlapping segments that are not ‘far’)
   If two segments are not touching, but too close for the thin wire kernel, the mutual impedance is calculated numerically using (23), with the appropriate integration limits.

### 3 Implementation

#### 3.1 Basis functions

Bilinear (rooftop) basis and testing functions, extending over two adjacent linear domains $\vec{h}_p$ and $\vec{h}_q$ (straight wire segments):

$$
\vec{h}_p(\alpha) = \vec{r}_k + \alpha(\vec{r}_l - \vec{r}_k), \quad 0 < \alpha < 1,
$$

(24)

and

$$
\vec{h}_q(\alpha) = \vec{r}_l + \alpha(\vec{r}_m - \vec{r}_l), \quad 0 < \alpha < 1,
$$

(25)

are defined as:

$$
\vec{b}_n(\vec{r}) = \begin{cases} 
\frac{\vec{r} - \vec{r}_l}{\Delta_p} & \text{on } \vec{h}_p (\vec{b}^-_n) \\
\frac{\vec{r} - \vec{r}_m}{\Delta_q} & \text{on } \vec{h}_q (\vec{b}^+_n) \\
0 & \text{elsewhere}
\end{cases}
$$

(26)

where $\Delta$ is the length of a segment

$$
\Delta_p = \Delta^-_p = |\vec{r}_k - \vec{r}_l|,
$$

(27)

$$
\Delta_q = \Delta^+_q = |\vec{r}_l - \vec{r}_m|.
$$

Other parameters associated with a segment are $\vec{u}$, a unit vector aligned with $\vec{b}_n$

$$
\vec{u}_p = \vec{u}^-_n = \frac{\vec{r}_l - \vec{r}_k}{\Delta_p},
$$

(28)

$$
\vec{u}_q = \vec{u}^+_n = \frac{\vec{r}_m - \vec{r}_l}{\Delta_q}.
$$
and $\vec{c}$, which points to the centre of the wire segment

$$\vec{c}_p = \vec{c}_{\pm n} = \frac{\vec{r}_k + \vec{r}_l}{2},$$

$$\vec{c}_q = \vec{c}_{\pm n} = \frac{\vec{r}_l + \vec{r}_m}{2}.$$

The divergence of the basis functions equals

$$\nabla \cdot \vec{b}_n = \frac{1}{\Delta p}, \quad \text{on } \vec{h}_p, \quad (30)$$

$$\nabla \cdot \vec{b}^+_n = -\frac{1}{\Delta q}, \quad \text{on } \vec{h}_q. \quad (31)$$

### 3.2 MoM impedance matrix

Testing of (1) with the basis functions, using the following inner product

$$\langle \vec{a}, \vec{b} \rangle = \int_{\vec{h}} \vec{a} \cdot \vec{b} \, dz \quad (31)$$

yields

$$\langle \vec{b}_m, \vec{E}^{inc} \rangle = \frac{jn}{4\pi k} \left\{ \langle \vec{b}_m, \vec{A} \rangle + \langle \vec{b}_m, \nabla \Phi \rangle \right\}. \quad (32)$$

(testing with the $m$-th basis function). This can be rewritten as ([5])

$$\langle \vec{b}_m, \vec{E}^{inc} \rangle = \frac{jn}{4\pi k} \left\{ \langle \vec{b}_m, \vec{A} \rangle - \int_{\vec{b}_m} \Phi \nabla \cdot \vec{b}_m \right\} \quad (33)$$

The impedance matrix elements $Z_{mn}$ are obtained by inserting into the RHS of (33) the current and charge represented by the $n$-th basis function, yielding

$$Z_{mn} = \frac{jn}{4\pi k} \sum_{n^\pm, m^\pm} \int_{h_m} \int_{h_n} \left\{ k^2 \left( \vec{b}^+_{n^\pm} \cdot \vec{b}^+_{m^\pm} \right) G(R) - \frac{G(R)}{\pm \Delta_n \Delta_m} \right\} dz^+_n dz^+_m \quad (34)$$

where the double integral is over the domains of basis and testing functions. The sum on the RHS of (34) is over the two segments of the basis and testing functions. The double integral is over the domain of the basis- and testing functions respectively. It can be seen that the formulation is perfectly symmetric with respect to the basis- and testing functions. However, this symmetry is broken in practice, because the two integrals are approximated in different ways. The inner integral, over the basis functions, is solved according to the methods explained in chapter 2. This is equivalent to evaluating the vector- and scalar potentials at given points in space. Those points are determined by the choice of integration rule (or approximation) for the second integral. Numerical experiments showed that the result converges if the vector potential is evaluated only at the joining node of the two segments making up a testing function (the
point $\vec{r}_i$ in (24) and (25)), and the scalar potential is evaluated at the points $\vec{c}_\pm$ in equation (29). Both are accurate up to $\delta(\Delta^2)$ (in the case of the vector potential the apriori knowledge is used that the integrand equals zero at the outer end points). Consequently, the integration rule also relies on $\Delta \ll \lambda$. The potentials for ‘far’ interactions (see chapter 2) are evaluated according to the same integration rules, such that the ‘far’ impedance matrix elements are perfectly symmetric. The ‘near’ interactions are not symmetric. However, in the wire implementation for FIESTA, it was chosen, in order to gain computational speed, to compute the value only once (with an arbitrary choice of the source- and field- wire, and use it both as $Z_{mn}$ and $Z_{nm}$. As the numerical results presented in chapter 4 show, the accuracy of the method is not compromised by this procedure.

### 3.3 Wires and RWG patches

To combine the wire code with the FIESTA RWG program, the interactions between wire segments and bi-triangular RWG basis functions must be determined. Since in both cases, the interaction involves the intermediary transformation to vector and scalar potentials, which are subsequently tested with (32) and Eq. (11) in [5] respectively, the corresponding impedance matrix elements are readily computed.

The ‘far’ interactions are perfectly symmetrical, so only the upper (or lower) half of the matrix needs to be computed. The ‘near’ interactions should also be symmetric in theory, since the EFIE is a symmetric operator, but in practise, the double integral is avoided. That is, the source potentials are determined through full integration, but they are evaluated at one field point only. This leads to two different approximations to the same value (interchanging the source- and field- wires). As explained in the previous section, FIESTA computes only one value for the two impedance matrix elements concerning pairs of wire rooftops. The same is done for the wire-patch interactions. Since the integration over the wires is much easier to implement and faster than that for the patches, the action of source wire A on field patch B is computed with an integration over the wire (according to chapter 2) and a field evaluation at the centroid of the patch, following [5]. Subsequently, this same value is used for the action of patch B on wire A. This technique is also validated with a numerical experiment in chapter 4.

### 4 Numerical Results

The code was validated against an established wire code, NEC [7]. The convergence and accuracy was tested with a half-wave dipole input impedance calculation. The wire is loaded with a delta gap voltage at the center and the input impedance is determined as $V/I$, where $I$ is the current on the central basis function. The results are shown in Figs 1 to 4. For very thin wire ($a = 10^{-19}\lambda$), the impedance converges at 256 unknowns both for FIESTA and
in NEC2. There is a small discrepancy between the two codes. This is probably due to the way the delta gap is defined. (the details are not given in the NEC2 manual). For the FIESTA simulation, $E_{\text{inc}}^{\text{inc}}$ is taken to equal $V/\delta$ over the gap of length $\delta$, centered at the center of the dipole, and zero everywhere else. The gap length was set to $\delta = 10^{-19} \lambda$.

For the $a = 10^{-4} \lambda$ dipole, the difference between the thin wire approximation and extended kernel is observed for both methods. The slow convergence is a well known problem for high accuracy wire MoM calculations like the input impedance.

The code was tested for bent wires with a monostatic RCS calculation for a circular loop antenna with loop-radius $b$ and wire radius $a$. The loop was modeled with 31 wire elements, both in FIESTA and in NEC2. The loop is positioned in the X-Y plane, and the incoming plane wave propagates along the X-axis, polarised along the Y-axis. Fig 5 shows the results. The correspondence between the methods is good.

The combined wire- and patch code was tested by comparing two ways to calculate the input impedance of a center-fed dipole close to a perfect ground. The dipole that was used was 0.1$\lambda$ in length, with a radius $a = 1 \times 10^{-19} \lambda$ and positioned parallel to the ground plane at a height $z = h$. It was modelled with 32 wire segments. First, using image theory, the ground plane was simulated by a second dipole, parallel to the first, at $z = -h$, and excited with an identical delta-gap pulse, except for the sign. This constituted the all-wire reference result. Second, the ground was simulated by a PEC square plate at $z = 0$, centered below the dipole, with dimensions $1\lambda \times 1\lambda$ and using 1160 RWG basis functions (average edgelength $0.057\lambda$). The results of the two simulations, as a function of $h/\lambda$ are shown in Figs. 6 and 7. It should be noted that the imaginary part of the input impedance is roughly a factor $10^4$ larger than the real part, so the imaginary part is more representative for the relative difference between the two computations. For larger values of $h$, there is a divergence visible in the graph of the real part. This is caused by the finite size of the PEC screen. The results correspond well between $h \approx 0.03\lambda$ and $h \approx 0.12\lambda$. Below that, they start diverging. Nevertheless, the relative error at $h/\lambda = 0.01$ still equals less than $1.5 \times 10^{-3}$. 


5 Conclusions

A summary is given of several approaches to the thin wire problem in computational electromagnetics, notably the approach taken in NEC, a well known Method of Moments package, and in FIESTA, the UPC/EEF in-house MoM package. The reasons for developing a wire functionality in FIESTA are specified: greater range of applicability and the possibility to use advanced solution methods such as MLMDA and MLFMA. Both arguments are of particular interest for the analysis of the electromagnetic properties of (pre-)fractal geometries. A theoretical and practical description of the wire implementation in FIESTA is followed by several numerical experiments validating it.
References


Figure 1: Input impedance of Delta-gap fed half-wave dipole, wire radius $10^{-19}$ $\lambda$.

Figure 2: Input impedance of Delta-gap fed half-wave dipole, wire radius $10^{-19}$ $\lambda$. 
Figure 3: Input impedance of Delta-gap fed half-wave dipole, wire radius $10^{-4} \lambda$.

Figure 4: Input impedance of Delta-gap fed half-wave dipole, wire radius $10^{-4} \lambda$. 
Figure 5: Circular loop antenna, loop radius $b$, wire radius $a$. Monostatic RCS in the plane of the loop against frequency.

Figure 6: Input impedance of Delta-gap fed $1/10 \lambda$ dipole, wire radius $10^{-19} \lambda$, at a height $h$ above a PEC ground.
Figure 7: Input impedance of Delta-gap fed 1/10 \( \lambda \) dipole, wire radius \( 10^{-19} \lambda \), at a height \( h \) above a PEC ground
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Software simulation tool

Formulation of numerical methods for fractal structures

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Abstract

This report describes some numerical approaches for solving EFIE of wire antenna analysis based on the Galërkin-Petrov expansion, suitable to be implemented for (pre)-fractal structure. The approach proposed proves to be very versatile and computationally efficient. Particular attention is oriented towards the analysis of the thin-wire approximation from the numerical point of view, by discussing critically its validity and pitfalls, in the light of the compactness property of the associated linear operator. A manifold of numerical simulations have been performed on different prefractal structures, highlighting some interesting properties of the current profiles.

Keywords: Numerical simulations, Wire antennas, Moment method, Galërkin approach, Thin-wire approximation, Spectral analysis
RELATED WP AND TASKS FROM THE PROJECT DESCRIPTION

This task within the FRACTALCOMS project is focused on providing the efficient simulation tools for analysing fractal wire antennas, and for discussing critically some delicate issues associated with the thin-wire approximation [1], that is on the approximation of a singular, albeit summable, kernel by means of a smooth (infinitely many times differentiable) one.

To quote from the FRACTALCOMS project:

The conventional numerical methods based on the Electric-Field Integral Equation (EFIE) must be reformulated for fractal structures. Also, the usual thin-wire approximation for the analysis of wire antennas are not valid when the wire is a fractal curve. The actions to do in this workpackage are:

• Use the results from task 2.2 to develop EFIE discrete formulation and numerical integration on fractal domains.
• Develop new approximations for thin-wire antennas along fractal curves.
• Compare results in a) and b) with classical numerical schemes pushed if possible to the limit of complexity afforded by the technology.

In developing this project we focused primarily on the development of the algorithms for an accurate simulation of fractal antennas, and on the analysis of the thin wire approximation (from the mathematical/computational point of view).

SECTION 1- Introduction

Albeit the Pocklington equation for dipolar antennas dates to 1897 [2], and very many articles and books have been written on wire antennas, the computational issues associated with the analysis of these radiating structures are, to a certain regard, still an open problem. This is confirmed by very recent articles published in 2001 and 2002 [3,4] on this issues.

Of course, the numerical problems are much more delicate in dealing with fractal wire antennas, since the geometric complexity of the structures is superimposed to the intrinsic difficulty of handling a singular integral equation.

The Task 3.2 ”Formulation of numerical methods for fractal structures” is essentially aimed at analyzing critically the problems associated with the numerical simulations of fractal structures based on the results achieved in Tasks 2.1 and 2.2, and at developing efficient computational approaches for a reliable numerical simulation of high-order iteration prefractal wire antennas.

2
This report described the results of these efforts in a succinct way, by focusing on the conceptual points underlying the development of numerical algorithms and numerically reliable approximations.

It is clear from the analysis developed in Workpackage 2 that any numerical simulation of the EFIE equations on fractal structures cannot be conveniently grounded on point-matching approaches, but should be framed within a weak formulation of the EFIE resulting from a Gal\'erkin-Petrov expansion [5], based on what customarily is referred to as a basis of entire-domain function spanning the space of square summable functions.

This approach has been seldom applied in the literature (albeit there are important exceptions, as for example the analysis of Yagi-Uda antenna configurations by Thiele [6]), and other techniques based on subdomain functions (piecewise linear, piecewise sinusoids, truncated cosines, etc.) have been preferred to it [7-10]. Moreover, some authors specifically criticize the use of entire-domain functions and the Gal\'erkin approach [11,12]. It is fairly easy to show that such a criticism by Sarkar is not grounded on any sound mathematical/computational problem, and the results developed in this report show clearly the practical applicability and convenience of the Gal\'erkin approach, which yields accurate results for the current profiles on prefactal wire antennas even for high iteration order in the construction process of the structure. Unless otherwise stated, we will make use of the wording “Gal\'erkin approach” to indicate the Gal\'erkin-Petrov expansion with respect to a system of entire-domain functions forming a basis for the space of square summable functions in the domain of interest.

Apart from the debate on the computational validity of the Gal\'erkin approach for the solution of the EFIE of wire antenna interest, there is another, very important problem, associated with the application of one of the classical tools of engineering (applied) electromagnetism, namely the thin-wire approximation [1]. Recent articles analyze this issue [3,4], concluding that the thin-wire approximation gives rise to an “ill-posed problem”, and that all the simulation based on this approach are scientifically unreliable.

Since the thin-wire approximation is one of most useful simplifying assumptions in applied electromagnetism, we have analyzed very carefully this issue, and we show that, albeit the criticism by Rynne et al. [4] and by Davies et al. [3] is motivated by a correct mathematical observation, in practical problems involving wire antennas possessing small radii compared to their length (this issue is quantitatively described in Section 2), the Gal\'erkin approach making use of the thin-wire approximation furnishes reliable and “convergent” approximations for the current profile and for all the electromagnetic quantity of physical and design interest.

This report is organized as follows. Section 2 analyzes the problem associated with the thin-wire approximation and the formulation of the Gal\'erkin approach for EFIE, by considering the dipolar antenna as a benchmark. Section 3 describes the formulation of the Gal\'erkin algorithm for tackling prefactal structures. Section 4 addresses succinctly some
significant numerical results out of the manifold of numerical simulations of fractal wire antennas carried out with the techniques described in Section 3. Finally the concluding section of this report summarizes the main achievements and the directions of future work.

SECTION 2- The good old boys: dipolar antennas

The dipolar antenna is the fundamental archetype of a radiating structure. In antenna theory and design, it plays the role of the harmonic oscillator in quantum mechanics, or the perfectly mixed stirred tank reactor in chemical reaction engineering.

Although the electromagnetic formulation of the EFIE equations describing dipolar antennas has been developed more than hundred years ago thanks to the work of Pocklington and subsequently of Hallén, there are still some computational issues that at the beginning of this new century are still unclear and await for a conclusive answer.

We use dipolar antennas in this report as a benchmark and as a guide for developing efficient numerical approaches to be applied for fractal wire antennas, and as a highlighting test-case in order to discuss some controversial issues associated with the thin-wire approximation.

This section is organized as follows. Section 2.1 describes the basic equations and the weak formulation of the Pocklington equation within the framework of the Galérkin expansion. Section 2.2 analyzes succinctly the criticism raised by Sarkar [11,12] on the Galérkin formulation of wire integral equations. Section 2.3 tackles the important issue of the thin wire approximation and its ill-posedness. Section 2.4 develops a spectral analysis of the Pocklington operator which gives an answer on the issue of the computational reliability of the thin-wire simplification.

2.1 Weak formulation and Galérkin expansion

Let us consider a dipolar symmetric wire antenna (made by a perfectly conducting material) of overall length \( L \) and radius \( a \), fed in its center. The EFIE describing its electromagnetic behaviour is the Pocklington equation:

\[
\left( k^2 + \frac{d^2}{d z'^2} \right) \int_{-L/2}^{L/2} I(z') G(z - z') dz' = -i 4 \pi \omega \varepsilon E_z^i(z)
\]  

(1)

where \( i = \sqrt{-1} \), \( I(z) \) is the unknown current on the antenna, \( E_z^i(z) \) the incident electric field, and \( G(z - z') \) the radiation kernel:

\[
G(z - z') = \frac{1}{2\pi} \int_{0}^{2\pi} e^{-ikR(z-z';\phi)} d\phi
\]  

(2)

\[
R(z - z';\phi) = \sqrt{(z - z')^2 + a^2}
\]
where
\[ R(z - z'; \phi) = \sqrt{4a^2 \sin^2(\phi/2) + (z - z')^2} \] (3)

The radiation kernel contains a summable singularity at \( z = z' \). A classical engineering approximation for getting rid of this singularity is to enforce the so-called thin-wire approximation. From the mathematical point of view, this implies to substitute the singular kernel Eq. (2), with the approximated one \( G_a(z - z') \):

\[ G_a(z - z') = \frac{e^{-ik R_a(z - z')}}{R_a(z - z')} , \quad R_a(z - z') = \sqrt{a^2 + (z - z')^2} \] (4)

so that Eq. (1) can be expressed in the thin-wire approximation as:

\[ \int_{-L/2}^{L/2} I(z') \left( k^2 + \frac{\partial^2}{\partial z'^2} \right) e^{-ik R_a(z - z')} \frac{d z'}{R_a(z - z')} = -i4\pi \varepsilon \varepsilon_0 \sqrt{\varepsilon / \mu} E_z^i(x) \] (5)

Eq. (5) can be made dimensionless by defining

\[ \theta = kL/2 \quad x = 2z/L , \quad y = 2z'/L , \quad \alpha = 2a/L , \quad \rho_a(z) = \sqrt{\alpha^2 + z^2} \] (6)

to obtain:

\[ \int_{-1}^{1} I(z') \left( \theta^2 + \frac{\partial^2}{\partial x^2} \right) e^{-ik \rho_a(z - y)} \frac{d y}{\rho_a(x - y)} = -i2\pi \theta \sqrt{\varepsilon / \mu} L E_z^i(x) \] (7)

Since by physical reasons \( I(\pm1) = 0 \), the current profile in a Galèrkin approach can be expanded in a system of function \( \{\phi_n(x)\} \) which forms a basis of the space \( L^2([-1, 1]) \) of square summable functions vanishing at the endpoints \( x = \pm1 \),

\[ I_n(x) = \sum_n I_n \phi_n(x) \] (8)

By choosing the same system of functions as trial functions, and by truncating the expansion up to the first \( N \) basis functions, one obtains a linear system for the coefficients \( \{I_n\}_{n=1}^{N} \)

\[ \sum_{m=1}^{N} A_{nm} I_m = f_n , \quad n = 1, 2, .., N \] (9)

where:

\[ A_{nm} = \int_{-1}^{1} \phi_n(x) \, dx \int_{-1}^{1} \phi_m(y) \left[ \theta^2 - \frac{\partial}{\partial x} \frac{\partial}{\partial y} \right] G_a(x - y) \, dy \] (10)

and

\[ f_n = -i2\pi \theta \sqrt{\varepsilon / \mu} L \int_{-1}^{1} E_z^i(x) \, dx \] (11)
which constitutes the representation of the weak formulation of the Pocklington equation.

Figure 1 shows a schematic representation of the dipolar antenna. If the feed is symmetrically placed, (for the feed can be chosen any arbitrary compactly supported function non identically vanishing for \(|x| \leq \beta \ll 1\)), a convenient system of basis function is given by

\[
\phi_n(x) = \cos \left[ \frac{(2n - 1)\pi x}{2} \right], \quad n = 1, 2, ..
\]

(12)

\[ y \]
\[ 1 \]
\[ \beta \]
\[ \text{Feed} \]
\[ -1 \]
\[ x \]

Figure 1: Schematic representation of a dipolar antenna.

By observing Eqs. (9)-(10), it follows immediately that two kinds of problems may occur in the numerical solution of this linear system of equations:

- the stiffness of the kernel for small \(\alpha\), which forces a delicate numerical integration for obtaining the coefficient matrix;
- the computational cost in performing the double-integral appearing in Eq. (10).

These two problems can be overcome by making use of sound elementary manipulations (which are conceptually analogous of those proposed by Werner et al. [13], and by Møsig [14] in connection with numerical computation of EFIE by means of subdomain functions).

First of all, the condition \(\phi_n(\pm 1)\) permits to transform Eq. (10) into:

\[
A_{nm} = \int_{-1}^{1} \int_{-1}^{1} \left[ \theta^2 \phi_n(x)\phi_m(y) - \phi_n'(x)\phi_m'(y) \right] G_\alpha(x - y) \, dx \, dy
\]

(13)
by a simple integration by part. In Eq. (13) we made use of the notation: $\phi'(x) = d\phi(x)/dx$.

The integration domain in Eq. (10) is given by

$$
\mathcal{D} = \{ (x, y) \mid -1 \leq x \leq 1, -1 \leq y \leq 1 \}
$$

The change of variable, $z = x - y$ transforms this integration domain into

$$
\mathcal{D}' = \{ (x, z) \mid -1 \leq x \leq 1, x - 1 \leq z \leq x + 1 \} = \mathcal{D}_1' \cup \mathcal{D}_2'
$$

(14)

where

$$
\mathcal{D}_1' = \{ (x, z) \mid 0 \leq z \leq 2, z - 1 \leq x \leq 1 \}
$$

$$
\mathcal{D}_2' = \{ (x, z) \mid 0 \leq z \leq 2, -1 \leq x \leq z + 1 \}
$$

(15)

as depicted in Fig. 2.

![Integration domain](image)

**Figure 2:** Integration domain of the Pocklington equation in the $x$-$z$ variables.

In this way, the estimate of the coefficient matrix $A_{nm}$ reduces to a single integral:

$$
A_{nm} = \int_0^2 G_a(z) \left[ \theta^2 L_{mn}(z) - M_{nm}(z) \right] \, dz
$$

(16)
where
\[
L_{nm}(z) = \int_{-1}^{1} \phi_n(x)\phi_m(x - z) \, dx + \int_{-1}^{1} \phi_n(x)\phi_m(x + z) \, dx \\
M_{nm}(z) = \int_{-1}^{1} \phi_n(x)\phi'_m(x - z) \, dx + \int_{-1}^{1} \phi_n(x)\phi'_m(x + z) \, dx
\]  
(17)

The integrals defining \( L_{nm}(z) \) and \( M_{nm}(z) \) can be straightforwardly estimated in closed forms for the basis functions Eq. (12):

\[
L_{nm}(z) = (2 - z) \delta_{nm} \cos \left[ \frac{(2n - 1)\pi z}{2} \right] \\
- \left[ \frac{(-1)^{n+m-1}}{\pi(n + m - 1)} + \frac{(-1)^{n-m}(1 - \delta_{nm})}{\pi(n - m)} \right] \sin \left[ \frac{(2n - 1)\pi z}{2} \right] \\
- \left[ \frac{(-1)^{n+m-1}}{\pi(n + m - 1)} - \frac{(-1)^{n-m}(1 - \delta_{nm})}{\pi(n - m)} \right] \sin \left[ \frac{(2m - 1)\pi z}{2} \right]
\]
(18)

\[
M_{nm}(z) = \frac{(2n - 1)(2m - 1)\pi^2}{4} \left\{ (2 - z) \delta_{nm} \cos \left[ \frac{(2n - 1)\pi z}{2} \right] \\
+ \left[ \frac{(-1)^{n+m-1}}{\pi(n + m - 1)} + \frac{(-1)^{n-m}(1 - \delta_{nm})}{\pi(n - m)} \right] \sin \left[ \frac{(2n - 1)\pi z}{2} \right] \\
+ \left[ \frac{(-1)^{n+m-1}}{\pi(n + m - 1)} - \frac{(-1)^{n-m}(1 - \delta_{nm})}{\pi(n - m)} \right] \sin \left[ \frac{(2m - 1)\pi z}{2} \right] \right\}
\]
(19)

where \( \delta_{nm} \) are the Kronecker symbols.

The second improvement to be performed consists in removing the stiffness of the kernel entering the definition of \( A_{nm} \):

\[
A_{nm} = \int_{0}^{\frac{\pi}{2}} \frac{\cos[\theta \rho_{a}(z)]}{\rho_{a}(z)} \left[ \theta^2 L_{nm}(z) - M_{nm}(z) \right] \, dz \\
- i \int_{0}^{\frac{\pi}{2}} \frac{\sin[\theta \rho_{a}(z)]}{\rho_{a}(z)} \left[ \theta^2 L_{nm}(z) - M_{nm}(z) \right] \, dz = A_{nm}^R - iA_{nm}^I
\]
(20)

While \( \sin(\theta \rho_{a}(z))/\rho_{a}(z) \) is a well behaved function in the neighbourhood of \( z = 0 \), the kernel of the real part is very steep in the neighbourhood of the origin. This problem can be overcome by integrating by parts. Since:

\[
\int \frac{dz}{\rho_{a}(z)} = \log(z + \sqrt{a^2 + z^2}) \quad \frac{d\rho_{a}(z)}{dz} = \frac{z}{\rho_{a}(z)}
\]
(21)

8
it follows that

\[
A_{nm}^R = \cos[\theta \rho_a(z)] \log(z + \sqrt{a^2 + z^2}) \left[\theta^2 L_{nm}(z) - M_{nm}(z)\right]_0^2 \\
- \int_0^2 \log(z + \sqrt{a^2 + z^2}) \cos[\theta \rho_a(z)] \left[\theta^2 L'_{nm}(z) - M_{nm}(z)\right] dz \\
+ \int_0^2 \theta z \sin[\theta \rho_a(z)] \log(z + \sqrt{a^2 + z^2}) \left[\theta^2 L_{nm}(z) - M_{nm}(z)\right] dz
\]  

(22)

The kernels entering Eqs. (20), (23) are regular and mild functions over the whole interval \( z \in [0,2] \), as can be observed from Fig. 3.

![Graph showing kernels](image)

Figure 3: Kernels entering the weak formulation of the Pocklington equation. \( \theta = 2\pi, \alpha = 10^{-2} \). a) \( f(z) = \sin[\theta \rho_a(z)]/\rho_a(z) \); b) \( \cos[\theta \rho_a(z)] \log(z + \sqrt{a^2 + z^2}) \); c) \( \theta z \sin[\theta \rho_a(z)] \log(z + \sqrt{a^2 + z^2})/\rho_a(z) \).

This simplifies significantly the estimate of these integrals by making use of a reasonable number of integration points for the same degree of numerical accuracy.

2.2 The Galéřkin expansion: reply to a criticism

In a series of articles [11,12], T. Sarkar criticizes the Galéřkin approach for solving EFIE equations, such as the Pocklington equation discussed throughout this section. To quote
this Author, the Galèrkin approach cannot be considered a useful method to this purpose, since “[the basis functions] cannot reconstruct the constant functions along the wire. Therefore, they cannot span the range of the operator (cfr. the Pocklington operator), and therefore should not be used as weighted functions. In other words, Galèrkin method should not be applied to the Pocklington’s equation” [12].

In point of fact, Sarkar’s work has significantly influenced the trend followed in computational electromagnetism for solving wire antenna problems. Since the observation of this Author lacks completely any sound mathematical basis, it is important to address and answer clearly and unambiguously to this criticism.

The Pocklington equation is a linear functional equation, which can be compactly expressed as

\[ A[I] = f \]  \hspace{1cm} (23)

the solution of which (the current profile \( I(z) \)) belongs to an infinite dimensional functional space. In approaching this equation, the first issue to be addressed is what we mean for solution, which essentially means to define the functional space in which the solution should be sought. The weak formulation of the Pocklington equation indicates that a proper functional space should be a subspace of the space of square summable functions \( L^2([-1, 1]) \). Consequently, the convergence of a numerical scheme towards the solution should be intrinsically viewed with respect to the norm of this functional space.

Moreover, it follows immediately from the definition, that the range of the Pocklington operator is a subspace of \( L^2([-1, 1]) \) (both in the case of the exact kernel and, \( a \) fortiori, in the case the thin wire approximation is enforced).

Since the solution (the current profile) should satisfy the physical condition at the endpoints

\[ I(\pm 1) = 0, \]  \hspace{1cm} (24)

any basis of \( L^2([-1, 1]) \) which satisfies these boundary conditions is a suitable candidate for a Galèrkin expansion of the wire antenna equations.

A (complete) basis of \( L^2([-1, 1]) \) can be obtained from the eigenfunctions of the Laplace operator

\[ \mathcal{L}[\phi] = \frac{d^2\phi(x)}{dx^2} \]  \hspace{1cm} (25)

in \( x \in [-1, 1] \) subjected to the boundary conditions

\[ \phi(x)|_{x=\pm 1} = 0. \]  \hspace{1cm} (26)

This is the basis which has been applied in order to solve the EFIE equations (with the further condition, induced by the symmetry of the problem considered, that solely even eigenfunctions contribute to the solution).
T. Sarkar claims further that such basis functions cannot be considered as valid test functions because a constant function cannot be expanded with respect to this basis. In point of fact, this observation is not supported by any sound argument, once it is interpreted within the metric of $L^2([-1, 1])$ (which is the only natural way to interpret the convergence of generalized Fourier series and weak solutions).

Consider the function $f(x) = 1, x \in [-1, 1]$, and the sequence of Fourier approximants:

$$f_N(x) = \sum_{n=1}^{N} a_n \cos \left[ \frac{(2n - 1)\pi x}{2} \right], \quad a_n = \frac{4(-1)^{n-1}}{(2n - 1)\pi}. \quad (27)$$

It is straightforward to prove that

$$||f(x) - f_N(x)||_{L^2} \to 0 \quad \text{for } N \to \infty,$$

as it should be expected in virtue of the fact that $\{\cos[(2n - 1)\pi x/2]\}$ is a basis in the subspace of even square summable functions on the interval $[-1, 1]$. Figure 4 depicts this fact, and shows the behaviour of $f_N(x)$ for several values of $N$.

There is a final and important observation on the $L^2$-convergence. The convergence in a $L^2$-sense, associated with the definition of weak solution, and intrinsically related to the Galërkin expansion, is fully grounded on physical reasons. All the electromagnetic quantities of interest in an antenna problem are expressed as weighted integrals of the current profile, so that any two current distributions which differ from each other within a set of zero Lebesgue measure would produce identical radiation properties.
Figure 4: $L^2$-approximations of the unit function in $[-1, 1]$ through the even basis $\{\cos[(2n-1)\pi x/2]\}$. (A) $N = 100$; (B) $N = 1000$; (C) $N = 10000$. 
2.3. Compactness and ill-posedness

While the Sarkar observation on the Galerkin expansion is arbitrary and unfair, as thoroughly discussed in the previous paragraph, there is a much more subtle and deep computational problem associated with the thin-wire approximation, which deserves attention and caution.

Let us consider again the Pocklington operator

$$A[I] = \left( \theta^2 + \frac{d^2}{dx^2} \right) \int_{-1}^{1} I(y) G(x - y) \, dy,$$

(29)

and its thin-wire counterpart

$$A_a[I] = \left( \theta^2 + \frac{d^2}{dx^2} \right) \int_{-1}^{1} I(y) G_a(x - y) \, dy,$$

(30)

The main difference between these two operators is that while the kernel $G(x - y)$ of the Pocklington operator possesses a summable singularity, the thin-wire kernel is smooth (i.e. it is infinitely many times differentiable). This property as a profound consequence, since it implies that the thin-wire Pocklington operator is compact. To prove it, it is sufficient to show that the image of any square summable function through the Pocklington operator is bounded (in the $L^2$ norm) and possesses bounded generalized derivatives (which follows straightforwardly from the smooth nature of the kernel).

The compact nature of the Pocklington operator admits a wealth of practical implications, since any compact linear operator possesses an eigenvalue spectrum which is upper-bounded in modulus, and the sequence of eigenvalues admits zero as an accumulation point. The latter properties is a source of numerical instability and makes the solution of the thin-wire Pocklington equation ill-posed.

This is concisely the logical reasoning developed by Rynne, Davies and coworkers, in a series of articles [3,4,15], containing a radical criticism to all the computational approaches based on the thin-wire approximation.

To quote these authors [4]:

"The concept of well-posedness is not an obscure mathematical idea that can be ignored, but has important consequences for numerical solutions. The above argument shows that large, highly oscillatory functions can be added to any given exact solution while only perturbing the right-hand side of eq. (1) (cfr. the Pocklington equation in the time-domain) by a small amount—say, less than some specified numerical tolerance or machine error. If the space and time mesh sizes in a numerical scheme are reduced enough for such oscillatory functions to be represented by the scheme then one would expect
these functions to appear and pollute numerical solution. This behaviour is indeed observed numerically.... Experienced practitioners may well have the skill and intuition to decide when the numerical results are good or bad, but this is not practical when for example a reduced kernel thin wire model is used inside a complicated code.”

and furthermore [3]:

“the ill-posedness” of the equation has significant implications for anyone trying to solve it numerically. Basically it means that \((1,1)\) (cfr. the Pocklington equation in the frequency domain) with the kernel \(K_{R}\) (cfr. the thin-wire kernel \(G_{a}\)) cannot give a “sensible” numerical solution unless the smallest space mesh-size \(h\) used in its discretization is large compared to the radius \(a\) of the wire. Taking \(h\) (cfr. the space mesh-size) to be too small results in the solution being polluted by an oscillation error near the ends of the wire that grows as \(h\) is reduced.. A consequence of this is that it is not possible to increase the accuracy of the approximation by refining the mesh, and indeed it is not clear what the computed “solution” actually represents. Although this has been known since the early 1950 (cfr. the book by Schelkunoff [16]), the reduced model still appears to be used in many situations.”

Such a strong and well motivated criticism on the thin-wire approximation requires careful and serious attention, since it is a clear scientific attack to an approximation so widely used within the engineering community. In order to understand more quantitatively this problem, next section addresses in detail the spectral analysis of the thin-wire Pocklington operator, in order to derive some concluding results that could be usefully applied in the development of numerical codes for fractal antennas.

2.4 Spectral analysis of the thin-wire approximation

The compactness of the Pocklington operator within the thin-wire approximation (henceforth referred to as the thin-wire Pocklington operator, TWPO) implies the ill-posedness of the design equation of wire antennas.

This problem does not automatically mean that any result based on this approximation is just “garbage” devoid of any scientific validity, but a clear computational analysis becomes necessary to ascertain the effects of the lack of well-posedness on the properties of the numerical solutions. Of course, this problem depends significantly on the computational/numerical approach followed, and this paragraph analyzes the Galërkin representation.

The compactness of a linear operator is essentially a spectral problem. Let \(A_{a}\) be the TWPO and \(A_{a}^{(N)}\) its \(N\)-th approximation by means of the Galërkin expansion, i.e. by
considering the first \( N \) basis functions. The TWPO can be expressed as follows:

\[
A_a = \sum_{i=1}^{N_s} \mu_i^s \langle p_i^s(x) \rangle + \sum_{j=1}^{\infty} \mu_j^u \langle p_j^u(x) \rangle
\]  

(31)

where \( \{\mu_i^s\} \) and \( \{\mu_j^u\} \) are respectively the families of (complex) eigenvalues possessing modulus larger \( (\mu_i^s) \) or smaller \( (\mu_j^u) \) than unity, while \( |p_i| \), \( \langle q_i \rangle \) are the right and left eigenvectors. In Eq. (31) we have made use of the Dirac’s bra-ket notation, which means that the action of \( A_a \) on a function \( \langle \phi(x) \rangle \) is given by

\[
A_a[\phi](x) = \sum_{i=1}^{N_s} \mu_i^s \langle q_i \rangle \langle p_i^s(x) \rangle + \sum_{j=1}^{\infty} \mu_j^u \langle q_j \rangle \langle p_j^u(x) \rangle
\]  

(32)

where \( \langle q_i \rangle \phi \) indicates the inner product.

The compactness of \( A_a \) implies that the family \( \{\mu_i^s\} \) is finite (and possibly void), while \( \{\mu_j^u\} \) is a numerable set of eigenvalues, the modulus of which admits zero as an accumulation point.

An analogous expression holds for \( A_a^{(N)} \),

\[
A_a = \sum_{i=1}^{N_s} \mu_i^{s,N} \langle q_i^{s,N}(x) \rangle + \sum_{j=1}^{N_u} \mu_j^{u,N} \langle q_j^{u,N}(x) \rangle
\]  

(33)

and the effects of the ill-posedness depend on the behaviour of the two families of eigenvalues \( \{\mu_i, N\}_{i=1}^{N_s}, \{\mu_j, N\}_{j=1}^{N_u} \) which \( N_s + N_u = 2N \) as a function of \( N \).

The estimate of eigenvalue spectrum for \( A_a^{(N)} \) reduced to the eigenvalue problem for the \( 2N \times 2N \) matrix

\[
\left( \begin{array}{cc} A_R & -A_I \\ A_I & A_R \end{array} \right)
\]  

(34)

where \( A_R \) and \( A_I \) are the matrix representations in the Galerkin basis \( \{\phi_i(x)\} \) of the real and imaginary part of the Pocklington operator.

In this problem, the dimensionless wave-vector \( \theta \) is a variable, and the dimensionless radius \( \alpha \) is the fundamental parameter controlling the physical validity of the thin-wire approximation (which, as the name itself states, is physically reasonable for small values of \( \alpha \)).

We have performed a detailed spectral analysis of the TWPO, the main results of which are depicted in Figs. 5-13.

Before commenting these results, it is important to observed that the spectrum of the Pocklington operator decouples into two branches: an even and an odd one, which correspond respectively to even and odd eigenfunctions of the position. Due to the symmetry
of the problem analyzed, solely the even branch is considered, and this corresponds in the Galërkin method to the expansion with respect to the even basis \{\cos((2n - 1)\pi z/2)\}.

Figure 5 shows the behaviour of the spectrum for “large radii”, \(\alpha = 0.1\), and for two different values of \(N\), \(N = 20\) and \(40\). This figure depicts the absolute value of the eigenvalues ordered in a non-decreasing sequence. It can be observed that while for \(N = 20\) all the eigenvalues possess a modulus larger than unity (i.e. no unstable eigenvalues are present\(^1\)), an increase in the number of modes (Fig. 5 (B), \(N = 40\)), produces the occurrence of an unstable family of eigenvalues (in modulus smaller than unity), which is responsible for the numerical instability in the numerical solution. These unstable eigenvalues are intrinsically related to the thin-wire approximation, and are the spectral fingerprint of the ill-posedness of the problem.

The dramatic effect of the unstable eigenvalue branch can be fully appreciated in Figs. 6 and 7 which depict respectively the current profiles for \(N = 40\) (Fig. 6), and the behaviour of the input resistance\(^2\) (Fig. 7) for several values of \(N\). The current profiles (the numerical value of the current is expressed in arbitrary units) display spurious oscillations (which has been observed near the endpoints also in numerical schemes based on a spatial discretization of the Pocklington equation \([10]\)), and the input resistance does not converges as a function of \(N\). Analogous results can be observed (although not explicitly reported for \(\alpha = 0.05\)).

Let us consider smaller radii. Figure 8 shows the behaviour of the eigenvalue spectrum for \(\alpha = 0.01\). Figure 8 (A) and 8 (B) refer respectively to two different values of \(\theta\), in order to show that the qualitative properties of the spectrum are independent of \(\theta\). Data refers to a modal expansion ranging from \(N = 20\) up to \(N = 160\). The comparison of Fig. 8 and Fig. 5 indicates deep qualitative differences, which can be summarizes as follows:

- Over a broad range of \(N\) (\(N = 20\text{-}160\)), the lower spectral limit (i.e. the smallest eigenvalue in modulus) is practically constant and definitely larger than unity.

- The whole spectrum display a “nested” behaviour, meaning with this wording that if we double the number of modes (say from \(N\) to \(2N\)), the eigenvalues occurring for \(N\) modes are present in the spectrum of the doubled Galërkin truncation \((2N)\).

\(^1\)Since we are considering an inverse problem, the wording “unstable” refers to the eigenvalues within the unit circle, while the “stable” eigenvalues possess modulus larger than unity.

\(^2\)The dimensionless quantity \(\theta/\pi\) equals the ratio \(L/\lambda\) of the dipole length to the wavelength \(\lambda\).
Figure 5: Spectrum of the thin-wire Pocklington operator (TWPO) for $\alpha = 0.1$ and $\theta = 0.1\pi$. (A) $N = 20$, (B) $N = 40$. 
Figure 6: Current profiles for $\alpha = 0.1$ and $N = 40$, for several values of $\theta$, starting from $\theta = 0.31416$ up to $\theta = 4.7123$.

Figure 7: Input resistance with $\theta$ for $a = 0.1$ (a) $N = 20$; (b) $N = 40$; (c) $n = 80$. 
Figure 8: Spectrum of the TWPO for $\alpha = 0.01$ (A) $\theta = 0.1\pi$; (B) $\theta = \pi$. Curves a) to d) refer respectively to $N = 20, 40, 80, 160$ modes.
These features are a clear indication that, at least for the considered range of $N$, the truncated TWPO does not display the numerical instabilities occurring for larger radii. This can be observed in Fig. 9, which depicts the current profiles at $\alpha = 0.01$ and in Fig. 10, showing the input impedance for several values of $N$. The current profiles do not show spurious high-frequency oscillations, and converge in the modal range considered ($N = 20$-$160$) towards a shape which is independent of $N$. The same convergent behaviour can be observed for the input impedance (Fig. 10). The input impedence obtained for $N = 80$ practically coincides with the same quantity estimated by doubling the number of modes.

The results are even better if one decreases the value of $\alpha$. Figures 11-13 show the eigenvalue spectrum, the current profiles and the input impedance for $\alpha = 10^{-3}$. In this case, the data for $N = 20$ (see Fig. 13) give a reliable approximation of the electromagnetic behaviour of the antenna.

Of course, if one considers an exceedingly large number of modes, the effects of compactness (and ill-posedness) may start to appear. But this is not the point. The main result spectral analysis has shown is that if one considers reasonable small radii (say $\alpha = 0.01$), Galërkin expansion of the TWPO gives reliable numerical results by considering a reasonable high number of modes in the expansion. Consequently, the criticism raised by Rynne, Davies and coworkers should be slightly tempered: if one consider sufficiently small radii it is still possible to use (with caution and attention) the thin-wire approximation in the analysis of dipolar antennas.

We conclude this paragraph by summarizing the main results achieved:

- Compactness of the TWPO is a serious and real problem affecting the computational analysis of wire antennas.
- For large radii ($\alpha \geq 0.01$), it affects in a dramatic way the numerical solutions obtained as it regards spurious oscillations and convergence towards a solution.
- The convergence properties of the input impedance are a significant lumped indicator of the reliability of a numerical scheme.
- The effects of the thin-wire approximation can be clearly analyzed by considering the spectral properties of the TWPO.
- For small radii ($\alpha \leq 10^{-2}$), the Galërkin expansion applied to the thin-wire reduced kernel, proves to be a reliable computational approach which gives acceptable numerical results.
Figure 9: Current profiles for $\alpha = 0.01$ for several values of $\theta$, starting from $\theta = 0.31416$ up to $\theta = 4.7123$. (A) $N = 20$; (B) $N = 40$, (C) $N = 80$. 
Figure 10: Input resistance (A) and input reactance (B) for \( \alpha = 0.01 \) The arrow indicates increasing values of \( N = 20, 40, 80, 160 \).
Figure 11: Spectrum of the TWPO for $\alpha = 10^{-3}$ at $\theta = \pi$. Curves a) to d) refer respectively to $N = 20, 40, 80, 160$. 
Figure 12: Current profiles for $\alpha = 10^{-3}$ for several values of $\theta$, starting from $\theta = 0.31416$ up to $\theta = 4.7123$. (A) $N = 20$; (B) $N = 40$, (C) $N = 80$. 

24
Figure 13: Input resistance (A) and input reactance (B) for $\alpha = 10^{-3}$ The arrow indicates increasing values of $N = 20, 40, 80$. 
SECTION 3- Galërkin expansion and fractal antennas

In this section we develop a Galërkin approach for analyzing prefractal antennas. By taking all the cautions (discussed in Section 2), the thin-wire approximation can be fruitfully applied and give acceptable and reliable approximations of the electromagnetic behaviour of a wire structure.

In approaching a generic curvilinear antenna, two characteristic lengthscales can be defined:

- $R_e$: the radius of the smallest sphere circumscribing the antenna;
- $L_a$: the length of the antenna itself.

It is convenient to express the EFIE in adimensionless form, by introducing a rescaling of the spatial coordinates so that the antenna is circumscribed (in the new coordinate system) within the unit sphere. Let

$$L = \frac{L_a}{R_e}, \quad \theta = kR_e,$$

and let $u \in [0, L]$ be the dimensionless curvilinear abscissa parametrizing the wire structure. By making use of this notation, the EFIE for a generic curvilinear wire antenna reads [17]:

$$\int_0^L I(v) \left[ \theta^2 \mathbf{t}(u) \cdot \mathbf{t}(v) - \frac{\partial}{\partial u} \frac{\partial}{\partial v} \right] G_a(u, v) = -i4\pi \omega \varepsilon R_e^2 E_1^2(u),$$

where $\mathbf{t}(u)$ is the tangent unit vector at the position $\mathbf{x}(u)$ corresponding to the value $u$ of the normalized curvilinear abscissa, $G_a(u, v)$ the thin-wire kernel:

$$G_a(u, v) = \frac{e^{-ik\rho_a(\mathbf{x}(u), \mathbf{x}(v))}}{\rho_a(\mathbf{x}(u), \mathbf{x}(v))},$$

$$\rho_a(\mathbf{x}(u), \mathbf{x}(v)) = \sqrt{\alpha^2 + ||\mathbf{x}(u) - \mathbf{x}(v)||^2},$$

and $\alpha = a/R_e$ is the dimensionless radius of the antennas.

It is useful to perform a further change of variables, by introducing the dimensionless normalized curvilinear abscissa $s$ defined as:

$$s = \frac{u}{L},$$

so that $s \in [0, 1]$. In this way it is possible to expand the current $I(s)$ along the wire antenna

$$I(s) = \sum_n I_n \phi_n(s),$$

26
with respect to a complete basis of square summable functions in \([0, 1]\), such as
\[
\phi_n(s) = \sin(n \pi s) \quad n = 1, 2, \ldots
\] (41)

Throughout this report we consider symmetric antenna configurations, in the presence of a symmetric feed:
\[
E^i(s) = E_{i0}^i \eta(s; s_1, s_2) = \begin{cases} 
E_{i0}^i & s \in [s_1, s_2] \\
0 & \text{otherwise} 
\end{cases}
\] (42)

where \(0 < s_1 < 1/2\), \(s_2 = 1 - s_1\), and \(E_{i0}^i\) is an arbitrary constant value, so that the current profile can be expanded with respect to the system of functions
\[
\phi_n(s) = \sin[(2n - 1)\pi s]
\] (43)

which is symmetric around \(s = 1/2\).

In this way, the weak formulation of the generalized Pocklington equation for curvilinear antennas reads:
\[
\sum_{m=1}^{N_m} I_m \int_0^1 \phi_n(s) \, ds \int_0^1 \phi_m(\sigma) \left[ \theta^2 \mathbf{t}(s) \cdot \mathbf{t}(\sigma) - \frac{1}{L^2} \frac{\partial}{\partial s} \frac{\partial}{\partial \sigma} \right] G_a(s, \sigma) \, d\sigma \\
= -i \int_0^1 \eta(s; s_1, s_2) \phi_n(s) \, ds
\] (44)

It is important to observe, that in Eq. (44) we have set the value of the constant \(E_{i0}^i\) in such a way the prefactor at the right-hand side equals 1. With this position, the input impedance is expressed by the equation:
\[
Z_{in} = \frac{60 \delta_s L^2}{\theta I(s_c)}
\] (45)

where \(\delta_s = (s_2 - s_1)/2 \ll 1\), and \(s_c = 1/2\).

Let us apply Eq. (45) to prefractional structures. In this case, the antenna is represented by a Lipschitz curve, formed by \(N_p\) segments specified by the values of the normalized curvilinear abscissa \(\{s_i\}_{i=0}^{N_p}\), \(s_0 = 1\), \(s_{N_p} = 1\) of the endpoints of each segment, the coordinates of which are \(\{\mathbf{x}_i\}_{i=0}^{N_p}\), where \(\mathbf{x}_i = (x_i, y_i)\), see Fig. 14. By following the piecewise linear structure of the antenna, the coefficient matrix can be expressed as:
\[
A_{nm} = \sum_{i=1}^{N_p} \sum_{j=1}^{N_p} \int_{s_{i-1}}^{s_i} \phi_n(s) \, ds \int_{s_{j-1}}^{s_j} \phi_m(\sigma) \left[ \theta^2 \mathbf{t}_i \cdot \mathbf{t}_j - \frac{1}{L^2} \frac{\partial}{\partial s} \frac{\partial}{\partial \sigma} \right] e^{-i \theta_{\rho_{a,ij}}(s, \sigma)} \rho_{a,ij}(s, \sigma) \, d\sigma \\
= \sum_{i=1}^{N_p} \sum_{j=1}^{N_p} A_{nm,ij}
\] (46)
where \( \rho_{a,ij}(s, \sigma) \) is the local representation of the function \( \rho_a \) for pair of points belonging to the \( i \)-th and \( j \)-th segments, and \( \mathbf{t}_i \) is the constant tangent unit vector of the \( i \)-th segment.

As extensively discussed in Section 2, an efficient computational approach to the solution of the generalized Pocklington equation for wire antenna structures should:

- remove kernel stiffness: in order to simplify the numerical integration of the coefficient matrix;
- reduce the computational burden in the estimate of the double integrals.

In point of fact, the coordinate transformation used to tackle the coefficient matrix in the case of the dipole cannot be performed for generic prefractal structures, and other approaches should be sought. In any case, it is important to stress that the Galerkin approach possesses an intrinsic and significant advantage compared to all the methods based on local functions: the solution of the corresponding linear system for the coefficients \( I_n \) of the current profile is not critical since in most of the cases it is sufficient to consider \( N = 20-40 \) modes independently of the geometric complexity of the structure.

Let us make a further comment related to the geometric structure of the antennas and on the numerical integration of the element of the coefficient matrix. The minimum number of segment forming the prefractal structure is an intrinsic property of the antenna. Nevertheless, in order to make the numerical integration more accurate, each linear segment is subdivided into equal subsegment such that each subsegment would not be longer than a prescribed characteristic linear size \( \Delta x \), which, depending on the complexity of the structure has been chosen in the range \([10^{-4}, 10^{-2}]\). Therefore, the integer \( N_p \) entering Eq. (46), should be considered as the total number of subsegment, forming the structure, the length of which is less or equal than \( \Delta x \).
In order to take $\Delta x$ as large as possible, by controlling in any case the accuracy of the estimate of the coefficient matrix, it is convenient to remove the stiffness of the kernel. This can be done, by following an approach essentially analogous to that one developed in Section 2 for dipolar antennas.

For pair of segments, the thin-wire distance functions $\rho_{a,ij}(s, \sigma)$ can be expressed as:

$$
\rho_{a,ij}^2(s, \sigma) = a_{ij} + b_{ij}s + c_{ij}s^2 + d_{ij}\sigma + e_{ij}\sigma^2 + f_{ij}s\sigma
$$

(47)

where

\[
\begin{align*}
  a_{i,j} &= \alpha^2 + (x_j - x_i)^2 + (y_j - y_i)^2 \\
  b_{i,j} &= -2[\alpha x_{i,j}(x_j - x_i) + \alpha y_{i,j}(y_j - y_i)] \\
  c_{i,j} &= \alpha_x^2 + \alpha_y^2 \\
  d_{i,j} &= 2[\alpha x_{i,j}(x_j - x_i) + \alpha y_{i,j}(y_j - y_i)] \\
  e_{i,j} &= \alpha_x^2 + \alpha_y^2 \\
  f_{i,j} &= -2(\alpha x_{i,j} + \alpha y_{i,j})
\end{align*}
\]

(48)

and

\[
\begin{align*}
  \alpha_{x,i} &= \frac{x_i - x_{i-1}}{s_i - s_{i-1}}, & \alpha_{y,i} &= \frac{y_i - y_{i-1}}{s_i - s_{i-1}}.
\end{align*}
\]

(49)

Since:

$$
R_{ij}(s, \sigma) = \int \frac{d\sigma}{\rho_{a,ij}(s, \sigma)} = \frac{1}{\sqrt{c_{i,j}}} \log \left[ d_{i,j} + f_{i,j}s + 2e_{i,j}\sigma + 2\sqrt{e_{i,j}\rho_{a,ij}(s, \sigma)} \right] \tag{50}
$$

and

$$
\frac{\partial \rho_{a,ij}}{\partial \sigma} = \frac{d_{i,j} + 2e_{i,j}\sigma + f_{i,j}s}{2\rho_{a,ij}} = \frac{1}{2\rho_{a,ij}} \frac{\partial \rho_{a,ij}^2}{\partial \sigma} \tag{51}
$$

by integrating by parts, it is possible to rewrite the real and imaginary part of the coefficient matrix as:

$$
A_{nm,ij}^R = \int_{s_{i-1}}^{s_i} \left[ \theta^2 L_{nm}(s, \sigma) + \frac{M_{nm}(s, \sigma)}{L^2} \right] \cos[\theta \rho_{a,ij}(s, \sigma) R_{ij}(s, \sigma)] ds d\sigma \tag{52}
$$

$$
A_{nm,ij}^I = -\theta \int_{s_{i-1}}^{s_i} \left[ \theta^2 L_{nm}(s, \sigma) + \frac{M_{nm}(s, \sigma)}{L^2} \right] \sin[\theta \rho_{a,ij}(s, \sigma)] \frac{1}{\theta \rho_{a,ij}(s, \sigma)} \frac{\partial \rho_{a,ij}^2}{\partial \sigma} ds d\sigma \tag{53}
$$

29
where

\[
L_{nm}(s, \sigma) = \mathbf{t}_i \cdot \mathbf{t}_j \phi_n(s) \phi_m(\sigma) \\
M_{nm}(s, \sigma) = -\phi_n'(s) \phi_m'(\sigma)
\]

\[
L'_{nm}(s, \sigma) = \mathbf{t}_i \cdot \mathbf{t}_j \phi_n(s) \phi'_m(\sigma) \\
M'_{nm}(s, \sigma) = -\phi_n'(s) \phi''_m(\sigma)
\]  

(54)

The second issue is to reduce the computational burden in the estimate of the double integrals. To this purpose, it is convenient to expand the functions \(\sin[\theta \rho_{a,ij}]/(\theta \rho_{a,ij})\) and \(\cos[\theta \rho_{a,ij}]\) in Taylor series:

\[
\sin[\theta \rho_{a,ij}] = \sum_{h=0}^{N_h} S_h \theta^{2h} \rho_{a,ij}^{2h} \quad S_h = \frac{(-1)^h}{(2h+1)!}
\]

\[
\cos[\theta \rho_{a,ij}] = \sum_{h=0}^{N_h} C_h \theta^{2h} \rho_{a,ij}^{2h} \quad C_h = \frac{(-1)^h}{(2h)!}
\]  

(55)

In point of fact \(N_h = 18-20\) Taylor terms are fully sufficient for an accurate reproduction of these functions for \(\theta\) in the range \([0, 2\pi]\). This is depicted in Fig. ??.

![Graph](image_url)

Figure 15: (a) \(f(x) = \sin[\theta x]/(\theta x)\), (b) \(f(x) = \cos[\theta x]\). Accuracy of the Taylor series expansion with \(N_h = 20\) for \(\theta = 2\pi\). Curves are the series expansions, dots the original functions.

The advantage of the series expansion is clearly evident. The coefficient matrix is reduced to a Taylor series expansion with respect to \(\theta\), the coefficient of which are integrals of the
form:

$$
\sum_{i=1}^{N_p} \sum_{j=1}^{N_p} \int_{s_{i-1}}^{s_i} \int_{s_{j-1}}^{s_j} f(s, \sigma) \rho^{2h}_{a,ij}(s, \sigma) \, ds \, d\sigma 
$$

(56)

where $f(s, \sigma)$ is a smooth function. One needs to estimate these coefficients once for all (the integration is performed by interpolating the functions $f(s, \sigma)$ and $\rho^{2h}_{a,ij}(s, \sigma)$ in a bilinear way), and subsequently by putting together all the terms one may calculate in a very efficient way the electromagnetic quantities for arbitrarily many values of $\theta$ in the range $[0, 2\pi]$. In this way, it is overcome the need of estimating the coefficient matrix for each value of $\theta$. Next section describes some numerical results obtained by applying this approach in the case of prefractal structures.

SECTION 4- Numerical results

This section analyzes some numerical results obtained by applying the method described in Section 3 to prefractal structures. We consider symmetric antenna structures, such as the 4-map Koch fractal antennas depicted in Fig. 16. Figure 18 shows the behaviour of

![Figure 16: 4-map Koch antenna. The fifth iterate is depicted.](image)

the input impedance for $\alpha = 0.01$. As expected, the input impedance convergences as the iteration increases towards an invariant shape. In point of fact, the convergence is controlled by the ratio $\alpha/l_c$, where $l_c$ is the smallest lengthscale of the structure (length of the smallest segment forming the prefractal structure). For smaller radii the convergence towards an invariant shape is attained at a higher iteration order in the construction
process of the structure, as depicted in Figs. 19-20. Qualitative analogous results have been obtained for other fractal structures. For example, Fig. 17 depicts the symmetric 8-map fractal antenna, and Fig. 21 the input impedance for $\alpha = 10^{-3}$ of this structure.

A significant feature of the Galerkin expansion is that it yields accurate current profiles, by considering a reasonable small amount of modes. The series expansion of the coefficient matrix permits to obtain, at a very “cheap” computational cost, these profiles over an arbitrarily large number of $\theta$-values. The three-dimensional current profiles, for a dipolar antenna, and for the fractal 4-map and 8-map wire antennas are depicted in Figs. 22-27. In these figures $I_{Rc}$ and $I_{Im}$ are the real and imaginary part of the current, respectively. The intensity of the current is normalized in such a way, that its maximum value in the range of $\theta$-values considered equals 1.

Two main results can be inferred from the analysis of these data:

- After few ($n = 3, 4$) iterations in the construction process of the structure, the current profile attains an almost invariant shape.

- The current profiles occurring in the 4-map and 8-map Koch wire antenna differ very little from a sinusoidal one.

![8-map Koch antenna. The third iterate is depicted.](image)

Figure 17: 8-map Koch antenna. The third iterate is depicted.
Figure 18: Input resistance (a) and input reactance (b) vs $\theta$ for the 4-map Koch wire antenna $\alpha = 10^{-2}$. The number of symmetric modes is $N = 20$. The arrow indicates increasing values of the iteration from 1 to 5.
Figure 19: Input resistance (a) and input reactance (b) vs $\theta$ for the 4-map Koch wire antenna $\alpha = 10^{-3}$. The number of symmetric modes is $N = 20$. The arrow indicates increasing values of the iteration from 1 to 5.
Figure 20: Input resistance (a) and input reactance (b) vs $\theta$ for the 4-map Koch wire antenna $\alpha = 10^{-5}$. The number of symmetric modes is $N = 20$. The arrow indicates increasing values of the iteration from 1 to 5.
Figure 21: Input resistance (a) and input reactance (b) vs $\theta$ for the 8-map Koch wire antenna $\alpha = 10^{-3}$. The number of symmetric modes is $N = 40$. The arrow indicates increasing values of the iteration from 1 to 3.
Figure 22: Dipole $\alpha = 10^{-3}$. $I_{Re}$ vs $\theta$-$s$. 
Figure 23: Dipole $\alpha = 10^{-3}$. $I_m$ vs $\theta$-$s$. 
Figure 24: 4-map Koch wire antenna. $n = 4$, $\alpha = 10^{-3}$. $I_{Re}$ vs $\theta$-$s$. 
Figure 25: 4-map Koch wire antenna, $n = 4$, $\alpha = 10^{-3}$. $I_{Im}$ vs $\theta-s$. 
Figure 26: 4-map Koch wire antenna, $n = 5, \alpha = 10^{-3}$. $I_{Re}$ and $I_{Im}$ vs $\theta$-s.
Figure 27: 8-map Koch wire antenna, $n = 3$, $\alpha = 10^{-3}$. $I_{lm}$ vs $\theta$-s.
More complex current distributions occur for wire structures which are wire-counterparts of fractal sets which “fill more densely the space”. To give an example, Fig. 28 shows the third iterate of a Sierpinski wire antenna. The structure is strictly a wire antenna in the meaning that there is neither overlapping or just-touching of the wire backbone, and it has been obtained with a slight modification of the IFS generating the Sierpinski gasket. Figures 29 and 30 show the current profile in this structures. It is evident from

![Wire Sierpinski antenna](image)

Figure 28: Wire Sierpinski antenna. The third iterate is depicted.

Fig. 30 the deviation for a sinusoidal behaviour occurring in these structures which is a consequence of the electromagnetic interaction between contiguous, but topologically non-adjacent parts of this antenna.

A detailed physical analysis of the results depicted in this section is outside the aim of this report, which describes essentially computational problems and simulation tools. What is important to stress is that the Galèrkin approach, coupled with the computational strategies outlined in Section 3, provides a very valuable tool for a better and deeper understanding of fractal wire antennas.
Figure 29: Sierpinski wire antenna, $n = 3 \alpha = 10^{-3}$. $I_{Re}$ and $I_{Im}$ vs $\theta$-$s$. 
Figure 30: Sierpinski wire antenna, $n = 3 \alpha = 10^{-3}$. Spatial distribution of $I_{Re}$ at $\theta/pi = 0.1083$. 
SECTION 5- Concluding remarks

This report has shown that the Galerkin approach provides an extremely useful computational tool for approaching wire antenna simulations. The main advantage of this method lies in the fact that \( N = 20-40 \) modes are in most cases sufficient for an acceptable representation of the electromagnetic behavior of the structure. The numerical solution of the associated linear systems for the generalized Fourier coefficients is computationally un-expensive, while the main computational cost lies in the estimate of the double integrals defining the coefficient matrix. In order to reduce this cost, several numerical tricks and alternative strategies (such as the series expansion of the kernel) have been developed to reduce the computational time of the simulations.

Another important property of the Galerkin expansion is that it enables an accurate estimate of the spatial current profiles along the wire structure. Several examples of fractal antennas have been analysis, and the simulations show that for “strict wire structures” (i.e. for those antennas the backbone of which is strictly a curve in a topological meaning), the current profiles are substantially sinusoidal. Conversely, significant deviations from a sinusoidal behaviour has been observed for wire models of fractal structures which are not topologically one-dimensional (such as for the wire-Sierpinski gasket).

We have thoroughly analyzed the thin-wire approximation in a computational/numerical perspective. The main results of this analysis, based on the spectral characterization of the Pocklington operator have been summarized at the end of Section 2. Albeit the compact nature of the TWPO (which results in the ill-posedness of the associated linear problem) is a serious numerical issue, we have shown that reliable results can be obtained within the thin-wire approximation by enforcing Galerkin expansion for small-radius \( \alpha \leq 10^{-2} \) antennas.

To conclude, it is important to mention the directions of our future research in the field of computational tools for wire-antenna analysis:

- the use of the Galerkin expansion for developing efficient numerical codes in the presence of the exact radiation kernel;

- the investigation of other classes of basis and frames for the Galerkin expansion. Specifically, our interest is oriented towards the application of wavelet frames and multiresolution analysis on the unit interval.
References


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